Find the radius (and interval) of convergence for

1. Recall the geometric series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad |r| < 1$$

Let $a = 1$ and $r = x$

(a) Write the sum of the series:

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} = a + ar + ar^2 + \cdots$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = 1 + x + x^2 + \cdots$$

(b) How would this be related to $f(x) = \dfrac{1}{1-x}$?

$a = 1 \quad r = x$

$f(x) \doteq 1 + x^2 + x^3 + \cdots x^n + \cdots = \frac{1}{1-r}$

Converges if $x < 1$

(c) What is the interval of convergence?

$$\lim_{n\to\infty} \left| \frac{ax^{n+1}}{ax^n} \right| = |x| < 1$$

$$-1 < x < 1$$

(d) Adapt the series so it is centered at -1

$1 + (1+x) + (1+x)^2 + (1+x)^3 + \cdots + (1+x)^n + \cdots$

2. (a) Find a power Series for $f(x) = \dfrac{4}{x+2}$

Hint: write this in the form of $\dfrac{a}{1-r}$

$$\frac{4}{2+x} = \frac{\frac{4}{2}}{\frac{2}{2}+\frac{x}{2}} = \frac{2}{1+\frac{x}{2}} = \frac{a}{1-r}$$

OR

so $\boxed{\begin{array}{l} a = 2 \\ r = -\frac{x}{2} \end{array}}$

$$\frac{4}{x+2} = a + ar + ar^2 + \cdots + ar^n + \cdots$$

$$= 2 - \frac{2x}{1} + \frac{2x^2}{} - \frac{2x^3}{} + \cdots + \frac{2x^n}{}$$

$$2\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n \qquad = 2\left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots (-1)^n\frac{x^n}{2^n} - \cdots\right)$$

(b) What is the interval of convergence?

$$|r| = \left|\frac{-x}{2}\right| < 1$$

$$|-x| < 2$$

$$-2 < x < 2$$

3. (a) Find a power series for $f(x) = \dfrac{1}{x}$ centered at 1

Hint: write this in the form of $\dfrac{a}{1-r}$

$$\frac{1}{x} = \frac{1}{1-(-x+1)} = \frac{a}{1-r}$$

$a = 1$
$r = -x+1 = 1-x$

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \ldots$$

(b) What is the interval of convergence?

Converges when

$$|x-1| < 1$$
$$-1 < x-1 < 1$$
$$0 < x < 2$$

4. (a) Find a power series for $f(x) = \frac{3x-1}{x^2-1}$ centered at 0 $= \frac{A}{x-1} + \frac{B}{x+1}$

$3x-1 = A(x+1) + B(x-1)$

$2 = 2A, A=1$

$-4 = -2B, B=2$

Hint: Use partial fraction decomposition, then use the geometric series trick with both

$$\frac{3x-1}{x^2-1} = \frac{2}{x+1} + \frac{1}{x-1}$$

$$\frac{2}{1-(-x)} = 2\sum_{n=0}^{\infty}(-1)^n x^n = 2 - 2x + 2x^2 + \cdots$$, Converges $|x|<1$

$$\frac{-1}{1-x} = -\sum_{n=0}^{\infty} x^n = -1 - x - x^2 - x^3 + \cdots$$, converges $|x|<1$

$$\frac{3x-1}{x^2-1} = 2 - 1 - 2x - x + 2x^2 - x^2 + \cdots + [2(-1)^n - 1]x^n + \cdots$$

$$= 1 - 3x + x^2 - 3x^3 + x^4 - \cdots$$

(b) What is the interval of convergence?

Both of the parts converge on $-1 < x < 1$

(so will the sum of the parts)

5. (a) Find a power series for $f(x) = \ln x$ centered at 1

Hint: Use integration! Didn't we do a series for $\frac{1}{x}$ before? yes — see #3

$$\ln x = \int \frac{1}{x}dx + C$$

from #3

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 + (x-1)^3 + \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n (x-1)^n \quad \text{converges on } (0,2)$$

$$\ln x = \int \frac{1}{x}\,dx$$
$$= C + \int \sum_{n=0}^{\infty} (-1)^n (x-1)^n \,dx$$
$$= C + \sum_{n=0}^{\infty} \frac{(-1)(x-1)^{n+1}}{n+1}$$
$$\ln x = \frac{(x-1)}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$$
$$\ln x+1 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

(b) What is the interval of convergence?

as before the integration

$(0,2)$

6. Find a power series for $f(x) = \arctan x$ centered at 0

(a) Recall
$$\frac{d}{dy}(\arctan y) = \frac{1}{1+\cancel{y}} \qquad \frac{1}{1+x^2}$$
substitute $y = x^2$. Doesn't
$$f(x^2) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}? \quad = 1 - x^2 + x^3 + x^4 + \dots$$
Now integrate to find the series for $\arctan x$ Converges $|x^2|<1$ so converges on $(-1,1)$

$$\arctan x = \int \frac{1}{1+x^2}\,dx + C$$
$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad (\text{let } x=0,\ C=0)$$
$$= \sum \frac{(-1)^n x^{2n+1}}{2n+1}$$
$$= x + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} + \dots + \frac{(-1)^n x^{2n+1}}{2n+1} + \dots$$

(b) It can be shown that this series for $\arctan x$ converges for $x = \pm 1$. What is the series approximation for $\arctan 1$. Use your calc's sum(seq()) function to add a 100 or so terms. Is it close to $\frac{\pi}{4}$?

sum (seq ((-1)^(N)* 1^(2N+1)/(2N+1), N, 0, 100)

≈ 0.7878733503

$\frac{\pi}{4} \approx 0.7853981634$

$4(\Sigma) \approx 3.151493401$

try more than 998 terms

Answers:
2a. $2(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \dots)$
2b. Converges on the open interval $(-2, 2)$
3a. $1 - (x-1)^1 + (x-1)^2 - (x-1)^3 + \dots$
3b. Converges on the open interval $(0, 2)$
4a. $f(x) = \frac{3x-1}{x^2-1} = \frac{2}{x+1} + \frac{1}{x-1}$ so $1 - 3x + x^2 - 3x^3 + x^4 - 3x^5 + \dots$
4b. Converges on the open interval $(-1, 1)$
5a. To find C, let $x = 1$, so $C = 0$. $f(x) = \ln x = \frac{2}{x+1} + \frac{1}{x-1}$ so $\frac{(x-1)^1}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$
5b. Converges on the open interval $(0, 2]$ — see how integration can change the convergence at the end points?
6a. $f(x) = \arctan x = \frac{2}{x+1} + \frac{1}{x-1}$ so $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$
6b. Converges on the open interval $(-1, 1)$
6b. Converges very slowly....